

GROUPS WITH TARSKI NUMBER 5

GILI GOLAN

ABSTRACT. The Tarski number of a non-amenable group G is the minimal number of pieces in a paradoxical decomposition of G . Until now the only numbers which were known to be Tarski numbers of some groups were 4 and 6. We construct a group with Tarski number 5 and mention a related result for Tarski numbers of group actions.

1. INTRODUCTION

Recall the definition of a *paradoxical decomposition* of a group.

Definition 1.1. A group G admits a *paradoxical decomposition* if there exist positive integers m and n , disjoint subsets $P_1, \dots, P_m, Q_1, \dots, Q_n$ of G and subsets $S_1 = \{g_1, \dots, g_m\}$, $S_2 = \{h_1, \dots, h_n\}$ of G such that

$$(1.1) \quad G = \bigcup_{i=1}^m g_i P_i = \bigcup_{j=1}^n h_j Q_j.$$

The sets S_1, S_2 are called the *translating sets* of the paradoxical decomposition.

It is well known [6] that G admits a paradoxical decomposition if and only if it is non-amenable. The minimal possible value of $m + n$ in a paradoxical decomposition of G is called the *Tarski number* of G and denoted by $\mathcal{T}(G)$.

It is clear that for any paradoxical decomposition we must have $m \geq 2$ and $n \geq 2$, so the minimal possible value of Tarski number is 4. By a theorem of Jónsson and Dekker (see, for example, [4, Theorem 5.8.38]), $\mathcal{T}(G) = 4$ if and only if G contains a non-abelian free subgroup.

Recently it was proved that the set of Tarski numbers is infinite [3]. At the time, no specific number other than 4 was known to be a Tarski number. The first and only result of the kind appeared shortly afterwards in [2], where a group with Tarski number 6 was constructed. The main feature of the proof was the use of random spanning forests on Cayley graphs. We shall use similar techniques to construct a group with Tarski number 5.

The notion of paradoxical decompositions and Tarski numbers naturally extends to group actions (see, for example, [6]). In this more general setting the problem of determining whether a given number is a Tarski number can be completely resolved. Indeed, for every $n \geq 4$ it is possible to construct a faithful transitive group action with Tarski number n . The proof and related results about Tarski numbers of group actions will appear at a future paper

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2. GROUPS WITH TARSKI NUMBER 5

In what follows we shall make use of the following criterion which follows from Lemma 2.5 and Theorem 2.6 in [2].

Lemma 2.1. *Let G be a group and S_1, S_2 finite subsets of G . The following are equivalent.*

- (1) *G has a paradoxical decomposition with translating sets S_1, S_2 .*
- (2) *For any pair of finite subsets $A_1, A_2 \in G$, $|A_1 S_1 \cup A_2 S_2| \geq |A_1| + |A_2|$.*

Proposition 2.2. *Let G be a group generated by $S = \{a, b, c\}$ and assume that a is an element of infinite order. If $\text{cost}(G) \geq 2.5$ then G has a paradoxical decomposition with translating sets $S_1 = \{1, a\}, S_2 = \{1, b, c\}$.*

Proof. Fix a pair of finite subsets A_1, A_2 of G . By Lemma 2.1 it suffices to prove that $|A_1 S_1 \cup A_2 S_2| \geq |A_1| + |A_2|$. Let $\Gamma = \text{Cay}(G, S \cup S^{-1})$ be the right Cayley graph of G with respect to $S \cup S^{-1}$ considered as an unoriented graph without multiple edges. The key result we shall use is the theorem of Thom [5, Theorem 3] which asserts that there exists a G -invariant random spanning forest μ of Γ such that μ -a.s the forest contains all edges labeled by $a^{\pm 1}$ and the expected degree of a vertex in μ is at least $2\text{cost}(G \curvearrowright [0, 1]^E)$ where E is the set of edges of Γ and the action $G \curvearrowright [0, 1]^E$ is the natural action.

An immediate corollary is that there exists an ordinary forest \mathcal{F} on Γ (depending on A_2) such that all edges labeled by $a^{\pm 1}$ belong to \mathcal{F} and

$$(2.1) \quad \sum_{g \in A_2} \deg_{\mathcal{F}}(g) \geq 2\text{cost}(G \curvearrowright [0, 1]^E)|A_2| \geq 2\text{cost}(G)|A_2| \geq 5|A_2|.$$

Let E be the set of all directed edges (g, gs) such that $g \in A_2$, $s \in S \cup S^{-1}$ and the unoriented edge $\{g, gs\}$ lies in \mathcal{F} . Let E_1 be the subset of E consisting of all edges $(g, gs) \in E$ with $s \in S \setminus S^{-1}$. Note that $|E| \geq 5|A_2|$ by (2.1), and it is clear that $|E_1| \geq |E| - |S||A_2|$, so that $|E_1| \geq 2|A_2|$.

Since the sets $S \setminus S^{-1}$ and $(S \setminus S^{-1})^{-1}$ are disjoint, E_1 does not contain a pair of opposite edges. Also, the label of every edge in E_1 belongs to $\{a, b, c\}$. Thus, if E_2 denotes the set of edges $(g, gs) \in E_1$ such that $g \in A_2$ and $s \in \{b, c\}$, $|E_2| \geq |E_1| - |A_2| \geq |A_2|$.

Let E_3 be the set of directed edges (g, ga) for $g \in A_1$. Clearly, E_2 and E_3 are disjoint sets and $E_2 \cup E_3$ does not contain a pair of opposite edges. The endpoints of edges in $E_2 \cup E_3$ lie in the set $A_1 S_1 \cup A_2 S_2$. Let Λ be the unoriented graph with vertex set $A_1 S_1 \cup A_2 S_2$ and edge set $E_2 \cup E_3$ (with forgotten orientation). Then Λ is a subgraph of \mathcal{F} ; in particular Λ is a (finite) forest. Hence

$$|A_1 S_1 \cup A_2 S_2| = |V(\Lambda)| > |E(\Lambda)| = |E_3| + |E_2| \geq |A_1| + |A_2|,$$

as desired. \square

Theorem 2.3. *Let $F = \langle a, b, c \rangle$ be a free group of rank 3. Let $r_1, r_2, \dots \in \gamma_2 F$ be an enumeration of the elements of the derived subgroup of F . Let $R = \{r_i^{p^{n_i}}\}$ for some integer sequence n_1, n_2, \dots such that $\sum_i \frac{1}{p^{n_i}} \leq \frac{1}{2}$. Then for $G = \langle X | R \rangle$, $\mathcal{T}(G) = 5$.*

Proof. By [2, Theorem B.1] G has a quotient Q such that $\beta_1(Q) \geq 1.5$ where $\beta_1(Q)$ is the first L^2 -Betti number of Q . In addition, Q can be chosen so that the image of a in Q has infinite order. Indeed, as in the proof of [2, Theorem B.1], let $R_m = \{r_i^{p^{n_i}}\}_{i=1}^m$,

$G(m) = \langle X | R_m \rangle$ and $G(m)_p$ be the image of $G(m)$ in its pro- p completion. Then, if $G(m)_p = F/N_m$, by the argument in [2], for $Q = F/\bigcup_{m \in \mathbb{N}} N_m$, $\beta_1(Q) \geq 1.5$. Assume by contradiction that the image of a in Q has finite order. Then, for some $m \in \mathbb{N}$, the image of a in $G(m)_p$ is also of finite order. Let $Ab: F \rightarrow \mathbb{Z}^3$ be the abelianization homomorphism. Since $R_m \subseteq \gamma_2 F$, Ab induces a homomorphism $Ab: G(m) \rightarrow \mathbb{Z}^3$. Let $i_{\mathbb{Z}^3}: \mathbb{Z}^3 \rightarrow \widehat{\mathbb{Z}^3}_p$ and $i_{G(m)}: G(m) \rightarrow \widehat{G(m)}_p$ be the natural homomorphisms from \mathbb{Z}^3 and $G(m)$ to their pro- p completions. Then, $\varphi = i_{\mathbb{Z}^3} \circ Ab: G(m) \rightarrow \widehat{\mathbb{Z}^3}_p$ is a homomorphism from $G(m)$ to a pro- p group. Clearly, φ is continuous when $G(m)$ is equipped with the pro- p topology. Thus, it can be extended in a unique way to a homomorphism $\psi: \widehat{G(m)}_p \rightarrow \widehat{\mathbb{Z}^3}_p$ for which $\psi \circ i_{G(m)} = \varphi$. Since the image of a under φ has infinite order in $\widehat{\mathbb{Z}^3}_p$, $i_{G(m)}(a)$ must have an infinite order in $G(m)_p \subseteq \widehat{G(m)}_p$, a contradiction.

Now we are ready to prove the theorem. For a quotient Q with the properties mentioned above, $\text{cost}(Q) \geq \beta_1(Q) + 1 \geq 2.5$. Thus, Proposition 2.2 applied to Q implies that $\mathcal{T}(Q) \leq 5$. Since Q is a quotient of G , $\mathcal{T}(G) \leq \mathcal{T}(Q) \leq 5$ (see, for example, [4, Theorem 5.8.16]). Since G is torsion-by-abelian it doesn't contain any free non abelian subgroup. Hence, by the theorem of Jónsson and Dekker mentioned above $\mathcal{T}(G) \neq 4$. Therefore $\mathcal{T}(G) = 5$ as required. \square

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BAR-ILAN UNIVERSITY

E-mail address: gili.golan@math.biu.ac.il